Continuity

1. Consider the function $f(x)$ defined as follows:

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   Analyze the continuity of $f(x)$ at $x = 0$.

2. Consider the function $g(x)$ defined as follows:

$$g(x) = \frac{x-3}{x^2-9}$$

   Analyze the continuity of $g(x)$ at $x = 3$ and $x = -3$.

3. Consider the function $h(x)$ defined as follows:

$$h(x) = \frac{x-7}{x^3-x}$$

   Analyze the continuity of $h(x)$.

4. Consider the function $g(x)$ defined as follows:

$$g(x) = \frac{x-3}{x^2+x}$$

   Analyze the continuity of $g(x)$.

5. Consider the function $f(x)$ defined as follows:

$$f(x) = \begin{cases} \frac{16}{x^2} & \text{if } x \geq 2 \\ 3x-2 & \text{if } x < 2 \end{cases}$$

   Analyze the continuity of $f(x)$ at $x = 2$.

Solutions

For a function $f(x)$ to be continuous at a point a, three conditions must be met:

- $f(a)$ is defined.
- $\lim_{x\to a} f(x)$ exists.
- $\lim_{x\to a} f(x) = f(a)$.

1. Let's check these conditions for $f(x)$ at $x = 0$:

   1. $f(0)$ is defined:
   $$f(0) = 0$$
   2. $\lim_{x\to 0} f(x)$ exists:
   $$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x} = \pm\infty$$
   The limit does not exist.
   3. $\lim_{x\to 0} f(x) = f(0)$: Since the limit does not exist, this condition is not met.

   Conclusion: The function $f(x)$ is not continuous at $x = 0$.

2. Let's check these conditions for $g(x)$ at $x = 3$ and $x = -3$:

   1. The function is not defined at $x = 3$ and $x = -3$.
   2. $\lim_{x\to 3} g(x)$ and $\lim_{x\to -3} g(x)$:
   $$g(x) = \frac{x-3}{(x-3)(x+3)} = \frac{1}{x+3} \quad \text{for } x \neq 3, -3$$
   $$\lim_{x\to 3} g(x) = \frac{1}{6}$$
   $$\lim_{x\to -3} g(x) = \text{undefined}$$
   The limit does not exist in a finite sense at $x = -3$.
   3. $\lim_{x\to 3} g(x) = g(3)$ and $\lim_{x\to -3} g(x) = g(-3)$: Since $g(3)$ and $g(-3)$ are not defined, this condition is not met.

   Conclusion: The function $g(x)$ is not continuous at $x = 3$ and $x = -3$.

   Consider redefining $f(x)$ in such a way that the function approaches a finite value at $x = 3$:
   $$f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 3 \\ 1/6 & \text{if } x = 3 \end{cases}$$
   Now the function is continuous at $x = 3$

3. Let's check these conditions for $h(x)$:

   1. $h(a)$ is defined:
   $$h(x) = \frac{x-7}{x(x-1)(x+1)}$$
   The function has discontinuities at $x = 0$, $x = 1$, and $x = -1$ because the denominator becomes zero.

   Since $h(0)$, $h(1)$, and $h(-1)$ are not defined, the function $h(x)$ is not continuous at these points.

4. Let's check these conditions for $g(x)$:
   1. $g(a)$ is defined:
   $$g(x) = \frac{x-3}{x(x+1)}$$
   The function has discontinuities at $x = 0$ and $x = -1$ because the denominator becomes zero.
   Since $g(0)$ and $g(-1)$ are not defined, the function $g(x)$ is not continuous at these points.

5. Let's check these conditions for $f(x)$ at $x = 2$:
   1. $f(2)$ is defined:
   $$f(2) = \frac{16}{2^2} = 4$$
   2. $\lim_{x\to 2} f(x)$ exists:
   $$\lim_{x\to 2^-} f(x) = 3(2) - 2 = 4$$
   $$\lim_{x\to 2^+} f(x) = \frac{16}{2^2} = 4$$
   Both the left-hand and right-hand limits are equal to 4. Therefore, $\lim_{x\to 2} f(x) = 4$.
   3. $\lim_{x\to 2} f(x) = f(2)$:
   $$\lim_{x\to 2} f(x) = f(2)$$
   Conclusion: The function $f(x)$ is continuous at $x = 2$.