Duality

Find the Marshallian demand functions for goods x_1 and x_2 if the agent has the following Hicksian demand functions:

$$x_1^h(p_1, p_2, u) = \left(1 + \sqrt{\frac{p_2}{p_1}}\right) u$$

$$x_2^h(p_1, p_2, u) = \left(1 + \sqrt{\frac{p_1}{p_2}}\right) u$$

Solution

We find the expenditure function by inserting the Hicksian demands into the budget constraint:

$$e = p_1(1 + \sqrt{p_2/p_1})u + p_2(1 + \sqrt{p_1/p_2})u$$

Now we invert the expenditure function to find the indirect utility function:

$$e = (p_1(1 + \sqrt{p_2/p_1}) + p_2(1 + \sqrt{p_1/p_2}))u$$

$$e = (p_1 + \sqrt{p_2 p_1}) + (p_2 + \sqrt{p_1 p_2}))u$$

$$e = (p_1 + p_2 + 2\sqrt{p_2 p_1})u$$

$$v = \frac{m}{p_1 + p_2 + 2\sqrt{p_2 p_1}} = m(p_1 + p_2 + 2\sqrt{p_1 p_2})^{-1}$$

Now using Roy's identity we obtain the Marshallian demand functions:

$$v'_{p_1} = -m(p_1 + p_2 + 2\sqrt{p_1 p_2})^{-2}(1 + \sqrt{p_2}p_1^{-1/2})$$

$$v'_m = (p_1 + p_2 + 2\sqrt{p_1 p_2})^{-1}$$

By Roy's identity:

$$\frac{v'_{p_1}}{v'_m} = \frac{-m(p_1 + p_2 + 2\sqrt{p_1 p_2})^{-2}(1 + \sqrt{p_2}p_1^{-1/2})}{(p_1 + p_2 + 2\sqrt{p_1 p_2})^{-1}} = -m(p_1 + p_2 + 2\sqrt{p_1 p_2})^{-1}(1 + \sqrt{p_2}p_1^{-1/2}) = -x_1^m$$

Now we do the same for the Marshallian demand for good 2:

$$v'_{p_2} = -m(p_1 + p_2 + 2\sqrt{p_1 p_2})^{-2}(1 + \sqrt{p_1}p_2^{-1/2})$$

$$\frac{v'_{p_2}}{v'_m} = \frac{-m(p_1 + p_2 + 2\sqrt{p_1 p_2})^{-2}(1 + \sqrt{p_1}p_2^{-1/2})}{(p_1 + p_2 + 2\sqrt{p_1 p_2})^{-1}} = -m(p_1 + p_2 + 2\sqrt{p_1 p_2})^{-1}(1 + \sqrt{p_1}p_2^{-1/2}) = -x_2^m$$

So the Marshallian demands are:

$$x_1^m = \frac{m(1 + \sqrt{p_2/p_1})}{(p_1 + p_2 + 2\sqrt{p_1 p_2})}$$

$$x_2^m = \frac{m(1 + \sqrt{p_1/p_2})}{(p_1 + p_2 + 2\sqrt{p_1 p_2})}$$