Elasticities

Consider the following utility function $u(x_1, x_2) = ax_1 - \frac{1}{2}x_1^2 + x_2$, where $a > 0$. Assume that the prices of the goods are p_1, p_2 and the income is m.

1. Assuming the solution is internal, find the Marshallian demand for good 1.
2. Suppose that $p_2 = 1$ and $a = 1$. Obtain the expression for the price elasticity of demand for good 1.
3. For which values of p_1 is the price elasticity of demand unitary? For which values of p_1 is demand elastic? And inelastic?

Solution

1. Maximize the utility, setting up the Lagrangian:

$$L = ax_1 - \frac{1}{2}x_1^2 + x_2 + \lambda(m - p_1x_1 - p_2x_2)$$

$$L'_{x_1} = a - x_1 - p_1\lambda = 0$$

$$L'_{x_2} = 1 + p_2\lambda = 0$$

$$L'_{\lambda} = m - p_1x_1 - p_2x_2 = 0$$

I derive the value of λ from the first two equations:

$$\frac{a - x_1}{p_1} = \lambda$$

$$\frac{1}{p_2} = \lambda$$

Equating the λ values:

$$\frac{a - x_1}{p_1} = \frac{1}{p_2}$$

$$a - x_1 = \frac{p_1}{p_2}$$

$$x_1 = a - \frac{p_1}{p_2}$$

Insert this into the third condition:

$$m - p_1(a - \frac{p_1}{p_2}) - p_2x_2 = 0$$

$$m - p_1a + \frac{p_1^2}{p_2} - p_2x_2 = 0$$

$$x_2 = \frac{m}{p_2} - \frac{p_1a}{p_2} + \frac{p_1^2}{p_2^2}$$

Assuming the solution is internal, $\frac{p_1}{p_2} < a$. We know that $x_1 = a - \frac{p_1}{p_2}$.

2. The elasticity formula is as follows:

$$\frac{Ex_1}{Ep_1} = \frac{\partial x_1}{\partial p_1}\frac{p_1}{x_1} = -(\frac{1}{p_2})\frac{p_1}{a - \frac{p_1}{p_2}}$$

Substituting with the given values:

$$\frac{Ex_1}{Ep_1} = -\frac{p_1}{1 - p_1}$$

3. For the elasticity to be unitary:

$$\frac{p_1}{1 - p_1} = 1$$

$$p_1 = 1 - p_1$$

$$p_1 = \frac{1}{2}$$

For demand to be elastic, $p_1 > \frac{1}{2}$ and for it to be inelastic, $p_1 < \frac{1}{2}$.