Homogeneity

Definition of Homogeneity

A function $F : \mathbb{R}^n_+ \to \mathbb{R}$ is said to be *homogeneous of degree k* if, for all $\lambda > 0$ and all $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_+$, it satisfies

$$F(\lambda x_1, \dots, \lambda x_n) = \lambda^k F(x_1, \dots, x_n)$$

When $k = 0$, the function is said to be *homogeneous of degree zero*.

Examples

1. **Homogeneity of degree 1 (linear).** Let

$$F(x, y) = 3x + 5y$$

   Then, for any $\lambda > 0$,

$$F(\lambda x, \lambda y) = 3(\lambda x) + 5(\lambda y) = \lambda (3x + 5y) = \lambda^1 F(x, y)$$

   Therefore, F is homogeneous of degree $k = 1$.

2. **Homogeneity of degree 0 (price ratio).** Let

$$G(x, y) = \frac{x}{y}$$

   For $\lambda > 0$,

$$G(\lambda x, \lambda y) = \frac{\lambda x}{\lambda y} = \frac{x}{y} = \lambda^0 G(x, y)$$

   Thus, G is homogeneous of degree $k = 0$.

Properties of Homogeneous Functions

1. **Scaling:** If F is homogeneous of degree k, then for all $\lambda, \mu > 0$

$$F(\lambda\mu\,\mathbf{x}) = (\lambda\mu)^k F(\mathbf{x}) = \lambda^k F(\mu\,\mathbf{x})$$

2. **Linear combination:** If F and G are homogeneous of degree k, and $a, b \in \mathbb{R}$, then

$$H(\mathbf{x}) = a\,F(\mathbf{x}) + b\,G(\mathbf{x})$$

   is homogeneous of degree k

3. **Product:** If F is homogeneous of degree k and G of degree m, then

$$(F \cdot G)(\mathbf{x}) = F(\mathbf{x})\,G(\mathbf{x})$$

   is homogeneous of degree $k + m$

4. **Quotient:** If $G(\mathbf{x}) \neq 0$ and F, G are homogeneous of degree k and m respectively, then

$$\left(\frac{F}{G}\right)(\mathbf{x}) = \frac{F(\mathbf{x})}{G(\mathbf{x})}$$

   is homogeneous of degree $k - m$

5. **Derivatives:** If F is homogeneous of degree k, then all its partial derivatives of order m are homogeneous of degree $k - m$

Euler's Theorem for Homogeneous Functions

Theorem (Euler). Let $F : \mathbb{R}^n_+ \to \mathbb{R}$ be a differentiable function that is homogeneous of degree k. Then,

$$\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}(x_1, \ldots, x_n) = k\, F(x_1, \ldots, x_n)$$

Proof. Since F is homogeneous of degree k, for all $\lambda > 0$ we have

$$F(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^k\, F(x_1, x_2, \ldots, x_n)$$

Define the single-variable function

$$\Phi(\lambda) \;=\; F(\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

By homogeneity, we also have

$$\Phi(\lambda) \;=\; \lambda^k\, F(x_1, \ldots, x_n)$$

Since F is differentiable, Φ is differentiable and we can differentiate both expressions with respect to λ:

- By the chain rule,

$$\Phi'(\lambda) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\lambda x_1, \ldots, \lambda x_n)\, \frac{d}{d\lambda}(\lambda x_i) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\lambda x_1, \ldots, \lambda x_n)\, x_i$$

- By differentiating $\lambda^k F(x_1, \ldots, x_n)$,

$$\Phi'(\lambda) = \frac{d}{d\lambda}(\lambda^k\, F(x_1, \ldots, x_n)) = k\, \lambda^{k-1}\, F(x_1, \ldots, x_n)$$

  Equating both expressions for $\Phi'(\lambda)$,

$$\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}(\lambda x_1, \ldots, \lambda x_n) = k\, \lambda^{k-1}\, F(x_1, \ldots, x_n)$$

Finally, evaluate at $\lambda = 1$:

$$\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i}(x_1, \ldots, x_n) = k\, F(x_1, \ldots, x_n)$$

as we wanted to prove. ■

Example

Consider the polynomial function

$$F(x, y) \;=\; x^3 \;+\; 2\, x\, y^2$$

1. **Verification of homogeneity.** Each term is of degree 3:

$$(\lambda x)^3 = \lambda^3 x^3, \qquad 2\,(\lambda x)\,(\lambda y)^2 = 2\,\lambda^3\, x\, y^2$$

   Therefore,

$$F(\lambda x, \lambda y) = (\lambda x)^3 + 2\,(\lambda x)\,(\lambda y)^2 = \lambda^3 (x^3 + 2\, x\, y^2) = \lambda^3\, F(x, y)$$

   and we conclude that F is homogeneous of degree $k = 3$.

2. **Partial derivatives.**

$$F_x(x, y) = \frac{\partial}{\partial x}(x^3 + 2\, x\, y^2) = 3x^2 + 2\, y^2$$

$$F_y(x, y) = \frac{\partial}{\partial y}(x^3 + 2\, x\, y^2) = 4\, x\, y$$

3. **Verification of Euler's Theorem.** Euler's theorem states that, if F is homogeneous of degree 3, then
$$x\,F_x(x,y) \;+\; y\,F_y(x,y) = 3\,F(x,y)$$
Compute the left-hand side:
$$x(3x^2+2y^2) \;+\; y(4xy) = 3x^3 + 2x\,y^2 + 4x\,y^2 = 3x^3 + 6x\,y^2$$
And the right-hand side:
$$3\,F(x,y) = 3(x^3+2x\,y^2) = 3x^3 + 6x\,y^2$$
Since both sides match,
$$x\,F_x + y\,F_y = 3\,F(x,y)$$
we confirm that Euler's theorem holds.

Economic interpretation

In economics, homogeneity and Euler's theorem have direct applications in production theory and demand analysis:

- **Production functions and returns to scale:** If the production function $Y = F(K,L)$ (capital K, labor L) is homogeneous of degree k:
$$F(\lambda K, \lambda L) = \lambda^k F(K,L),$$
then:
  - $k=1$ implies *constant returns to scale*: doubling all inputs doubles output.
  - $k>1$ implies *increasing returns to scale*: doubling inputs more than doubles output.
  - $k<1$ implies *decreasing returns to scale*: doubling inputs less than doubles output.
- **Demand homogeneous of degree zero in prices and income:** A demand function $x_i(p_1,\dots,p_n,M)$ (prices p_j, income M) is homogeneous of degree zero:
$$x_i(\lambda p_1,\dots,\lambda p_n,\lambda M) = x_i(p_1,\dots,p_n,M)$$
This means that if all prices and income change in the same proportion, demanded quantities remain unchanged: only relative prices and real purchasing power matter.
- **Partial elasticities and degree of homogeneity:** Let $F(K,L)$ be a differentiable production function that is homogeneous of degree k. Define the partial elasticities of output with respect to each factor as
$$\varepsilon_K = \frac{\partial F}{\partial K}\,\frac{K}{F(K,L)}, \qquad \varepsilon_L = \frac{\partial F}{\partial L}\,\frac{L}{F(K,L)}$$
Applying Euler's theorem:
$$K\,F_K(K,L) + L\,F_L(K,L) = k\,F(K,L),$$
and dividing both sides by $F(K,L)$, we obtain
$$\varepsilon_K + \varepsilon_L = k$$
***Interpretation:* the sum of the partial elasticities of output with respect to capital and labor equals the degree of homogeneity of the function. Economically, this means that returns to scale match the sum of the partial elasticities**
  - If $\varepsilon_K + \varepsilon_L = k = 1$, constant returns to scale.
  - If $\varepsilon_K + \varepsilon_L = k > 1$, increasing returns to scale.
  - If $\varepsilon_K + \varepsilon_L = k < 1$, decreasing returns to scale.