IS–LM

Introduction

The IS–LM model simultaneously determines real income Y and the interest rate r in the short run, through equilibrium in two markets: the goods and services market (IS curve) and the money market (LM curve). It also incorporates the relationship between income and tax revenue.

$$\begin{cases} Y = C(Y_d) + I(r) + g, \\ M_p = L(Y, r), \\ Y_d = Y - T(Y, \theta) \end{cases}$$

1. Goods and services market (IS)

$$Y = C(Y_d) + I(r) + g$$

- Y: aggregate output (or income)
- $C(Y_d)$: household consumption, an increasing function of disposable income Y_d
- $I(r)$: business investment, a decreasing function of the interest rate r
- g: government expenditure, exogenous

The IS curve ("Investment–Saving") is the set of (Y, r) pairs that satisfy this identity.

2. Money market (LM)

$$M_p = L(Y, r)$$

- M_p: real money supply (stock), set by the monetary authority
- $L(Y, r)$: money demand, increasing in Y ($\partial L/\partial Y > 0$) and decreasing in r ($\partial L/\partial r < 0$)

The LM curve ("Liquidity preference–Money supply") is the set of (Y, r) points where money supply equals money demand.

3. Disposable income

$$Y_d = Y - T(Y, \theta)$$

where $T(Y, \theta)$ is tax revenue, an increasing function of Y. Disposable income Y_d determines actual consumption.

Solving the IS–LM model by implicit differentiation

We start with the system:

$$\begin{cases} Y = C(Y_d) + I(r) + g, \\ M_p = L(Y, r), \\ Y_d = Y - T(Y, \theta) \end{cases}$$

Endogenous variables	
Y	Equilibrium income (output)
r	Equilibrium interest rate

Exogenous variables	
g	Government expenditure
M_p	Real money supply (monetary stock)
θ	Fiscal parameter(s) in $T(Y, \theta)$

Behavioral assumptions:

$$0 < C'(Y - T(Y,\theta)) < 1, \quad 0 < T_Y(Y,\theta) < 1, \quad I'(r) < 0, \quad L_Y(Y,r) > 0, \quad L_r(Y,r) < 0, \quad T_\theta(Y,\theta) > 0$$

Substituting Y_d into the first equation, we obtain two equations in the endogenous variables Y and r:

$$\begin{cases} F(Y, r; g, \theta) \;=\; Y - C(Y - T(Y,\theta)) - I(r) - g \;=\; 0, \\ G(Y, r; M_p) \;=\; L(Y, r) - M_p \;=\; 0 \end{cases}$$

The objective is to find $\dfrac{\partial Y}{\partial g}$ and $\dfrac{\partial r}{\partial g}$, that is, the effect of an increase in government expenditure on output and the interest rate.

Step 1: Partial derivatives

We differentiate both equations with respect to g. Recall that both Y and r depend on g, so we must apply the chain rule:

$$F'_g = \frac{\partial F}{\partial Y} \cdot \frac{\partial Y}{\partial g} + \frac{\partial F}{\partial r} \cdot \frac{\partial r}{\partial g} + \frac{\partial F}{\partial g} \cdot \frac{\partial g}{\partial g} = 0$$

$$G'_g = \frac{\partial G}{\partial Y} \cdot \frac{\partial Y}{\partial g} + \frac{\partial G}{\partial r} \cdot \frac{\partial r}{\partial g} + \frac{\partial G}{\partial g} \cdot \frac{\partial g}{\partial g} = 0$$

We now differentiate both equations with respect to g, keeping M_p and θ constant:

$$\frac{\partial}{\partial g}\Big[Y - C(Y - T(Y,\theta)) - I(r) - g\Big] = \Big(1 - C'(Y - T(Y,\theta))\,[1 - T_Y(Y,\theta)]\Big)\,\frac{\partial Y}{\partial g} - I'(r)\,\frac{\partial r}{\partial g} - 1 = 0,$$

$$\frac{\partial}{\partial g}[L(Y,r) - M_p] = L_Y(Y,r)\,\frac{\partial Y}{\partial g} + L_r(Y,r)\,\frac{\partial r}{\partial g} = 0.$$

Thus, the system of equations becomes:

$$\begin{cases} \Big(1 - C'(Y - T(Y,\theta))\,[1 - T_Y(Y,\theta)]\Big)\,\dfrac{\partial Y}{\partial g} - I'(r)\,\dfrac{\partial r}{\partial g} = 1, \\ L_Y(Y,r)\,\dfrac{\partial Y}{\partial g} + L_r(Y,r)\,\dfrac{\partial r}{\partial g} = 0 \end{cases}$$

This system can now be solved using Cramer's rule to obtain the fiscal multipliers $\frac{\partial Y}{\partial g}$ and $\frac{\partial r}{\partial g}$.

Step 2: Matrix form

This gives us the following linear system:

$$\begin{pmatrix} 1 - C'(Y - T(Y,\theta))\,[1 - T_Y(Y,\theta)] & -I'(r) \\ L_Y(Y,r) & L_r(Y,r) \end{pmatrix} \begin{pmatrix} \frac{\partial Y}{\partial g} \\ \frac{\partial r}{\partial g} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Step 3: Solve using Cramer's Rule

The determinant of the system is

$$J = [1 - C'(Y - T(Y,\theta))\,[1 - T_Y(Y,\theta)]] \cdot L_r(Y,r) - [-I'(r)] \cdot L_Y(Y,r)$$

By Cramer's rule,

$$\frac{\partial Y}{\partial g} = \frac{\begin{vmatrix} 1 & -I'(r) \\ 0 & L_r(Y,r) \end{vmatrix}}{J} = \frac{1 \cdot L_r(Y,r)}{J} = \frac{L_r(Y,r)}{J},$$

$$\frac{\partial r}{\partial g} = \frac{\begin{vmatrix} 1 - C'(Y - T(Y,\theta))\,[1 - T_Y(Y,\theta)] & 1 \\ L_Y(Y,r) & 0 \end{vmatrix}}{J} = \frac{-L_Y(Y,r)}{J}$$

Step 4: Signs and interpretation

Given that

$$0 < C'(Y - T(Y,\theta)) < 1, \quad 0 < T_Y(Y,\theta) < 1, \quad I'(r) < 0, \quad L_Y(Y,r) > 0, \quad L_r(Y,r) < 0,$$

it follows that

$$1 - C'(Y - T(Y,\theta))\,[1 - T_Y(Y,\theta)] > 0, \quad -I'(r) > 0$$

Therefore, the determinant is

$$J = \underbrace{\Big[1 - \overbrace{\underbrace{C'(Y - T(Y,\theta))}_{0<\cdot<1} \times \underbrace{(1 - T_Y(Y,\theta))}_{0<\cdot<1}}^{0<\cdot<1}\Big]}_{>0} \times \underbrace{L_r(Y,r)}_{<0} - \underbrace{(-I'(r))}_{>0} \times \underbrace{L_Y(Y,r)}_{>0} \; < \; 0$$

Since $L_r < 0$ and $L_Y > 0$, the numerator of $\partial Y/\partial g$ is negative and the numerator of $\partial r/\partial g$ is also negative:

$$\frac{\partial Y}{\partial g} = \frac{L_r}{J} > 0, \qquad \frac{\partial r}{\partial g} = \frac{-L_Y}{J} > 0$$

When the government increases its spending g, two main effects occur:

More demand ⇒ higher output (Y)

- *Direct spending*: the government injects money by paying for infrastructure, wages, inputs, etc.
- *Spending chain*: those payments become income for firms and workers, who then consume part of it, generating more sales and income elsewhere.
- *Multiplier effect*: each unit of public spending leads to more than one unit increase in aggregate income, as money circulates and is spent multiple times.

Higher Y ⇒ more demand for money ⇒ higher r

- *Transactional demand for money*: as Y increases, households and firms make more transactions and need more liquidity.
- *LM curve*: $L(Y, r)$ rises with Y and falls with r. With fixed M_p, an increase in money demand raises the interest rate r until the money market clears.
- *Partial crowding out*: as r rises, borrowing becomes more expensive, and private investment slows down, partially offsetting the initial boost to Y but not eliminating it.