Maximizing production with second order conditions and interpretation of the Lagrange multiplier

A production function is given by $q = \sqrt{x_1 x_2}$ where x_1 and x_2 are the quantities of the inputs whose prices are $p_1 = 1$ and $p_2 = 2$

1. Determine the quantities that maximize production if the total cost is 24.
2. Verify that the found point is a maximum from the bordered Hessian matrix.
3. Without solving the problem, estimate what the maximum production would be if the cost increases by one unit.

Solution

1.

$$L = \sqrt{x_1 x_2} + \lambda(24 - p_1 x_1 - p_2 x_2)$$

$$L' x_1 = \frac{1}{2}(x_1 x_2)^{-1/2} x_2 - \lambda p_1 = 0$$

$$L' x_2 = \frac{1}{2}(x_1 x_2)^{-1/2} x_1 - \lambda p_2 = 0$$

$$L' \lambda = 24 - p_1 x_1 - p_2 x_2 = 0$$

From the first two equations:

$$\frac{1}{2p_1}(x_1 x_2)^{-1/2} x_2 = \frac{1}{2p_2}(x_1 x_2)^{-1/2} x_1$$

Assuming that x_1 and x_2 are different from 0 (as we are looking to maximize we can discard the cases where x_1 or x_2 are equal to 0).

$$\frac{1}{p_1} x_2 = \frac{1}{p_2} x_1$$

$$\frac{p_2}{p_1} x_2 = x_1$$

We use this in the last condition:

$$24 - p_1 \frac{p_2}{p_1} x_2 - p_2 x_2 = 0$$

$$24 - p_2 x_2 - p_2 x_2 = 0$$

$$x_2 = \frac{12}{p_2}$$

With this we get that:

$$x_1 = \frac{12}{p_1}$$

2. To check the second order conditions, instead of working with the original function we apply a strictly increasing transformation, raising the objective function to 2, the result is: $f = x_1 x_2$. This can be done because the maximum or minimum points are maintained through strictly increasing transformations, the second derivatives leave us with the following bordered Hessian: $\bar{H} = \begin{pmatrix} 0 & p_1 & p_2 \\ p_1 & 0 & 1 \\ p_2 & 1 & 0 \end{pmatrix}$

   Whose determinant is:

$$|\bar{H}| = -p_1[-p_2] + p_2[p_1] = p_1 p_2 + p_2 p_1 = 2p_1 p_2 > 0$$

   Therefore, we are dealing with a maximum.

3. If we want to see how much the production would be if the cost increases by one unit, we look at the value of λ. Remember that λ at the optimum tells us how the maximum production changes if we relax the restriction (i.e., increase the available budget). At the optimum, this is:

$$\lambda = \frac{1}{p_1} \frac{12}{p_2} = \frac{12}{p_1 p_2}$$